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A Remark on Multivariate B-Splines*

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Generalizing an idea of I. J. Schoenberg [3], C. de Boor [1] introduced a geometric definition for multivariate B-splines. Recently C. A. Micchelli [10] proved the fundamental recurrence relations. Moreover in [11] he obtained many other striking multivariate identities. Independently W. Dahmen [4] generalized the one dimensional truncated powers and expressed the B-splines as linear combinations of the fundamental solutions of certain differential equations. This also led to recurrence relations. In their subsequent work W. Dahmen [5–7] and C. A. Micchelli [11, cf. also 2] further developed the theory of non-tensor product splines.

Here we obtain the basic recurrence relations for multivariate B-splines starting from the equivalent description by their Fourier transforms given in [10]. This should be a useful alternative to the more geometrically orientated approaches in [4, 10]. We have employed the notation used in [10] and unless stated otherwise all identities should be interpreted in the distributional sense.

Denote by $[t_0,..., t_n]f$ the divided dfference of a smooth function f. Since in the univariate case the B-spline $M(\cdot \mid z_0,..., z_n)$ is the Peano kernel of the divided difference we have

$$\hat{M}(y \mid z_0,..., z_n) = \int_{-\infty}^{\infty} M(x \mid z_0,..., z_n)\, e^{-ixy}\, dx$$

$$= [z_0,..., z_n]_x \frac{e^{-ixy}}{(-iy)^n} = [-iyz_0,..., -iyz_n]_t\, e^t.$$

As shown by Micchelli [10] this formula has a natural generalization to several dimensions which we will take as alternative definition of the multivariate B-spline.

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DEFINITION. We define the B-spline M corresponding to the knots $z_\nu \in \mathbb{R}^k$ by its Fourier transform

$$\hat{M}(y \mid z_0, \ldots, z_n) := [-iy \cdot z_0, \ldots, -iy \cdot z_n]_t \, e^t. \tag{1}$$

Hence along any ray sv, $s \in \mathbb{R}$, $v \in \mathbb{R}^k$, $\hat{M}(sv \mid z_0, \ldots, z_n)$ coincides with the Fourier transform of the univariate B-spline $M(\cdot \mid v \cdot z_0, \ldots, v \cdot z_n)$. The right hand side of (1) is an entire function of y. More precisely we have, by the Hermite–Genocchi formula, for $y = \xi + i\eta \in \mathbb{C}^k$,

$$|\hat{M}(y)| = \left| \int_{S^n} \exp \sum_{\nu=0}^{n} \lambda_\nu(-iy \cdot z_\nu) \, d\lambda_1 \cdots d\lambda_n \right| \leqslant \frac{1}{n!} e^{H(\eta)},$$

where $S^n = \{(\lambda_0, \ldots, \lambda_n) \mid \sum_{\nu=0}^{n} \lambda_\nu = 1, \ \lambda_\nu \geqslant 0\}$ denotes the standard n-dimensional simplex and $H(\eta) = \max_\nu e^{\eta \cdot z_\nu}$ is the support function of the convex hull $\operatorname{conv}\{z_0, \ldots, z_n\}$ of the knots z_ν. Therefore, by the Paley–Wiener theorem [8, p. 213], Eq. (1) defines M as a distribution with $\operatorname{supp} M \subseteq \operatorname{conv}\{z_0, \ldots, z_n\}$.

We write $\langle S, \phi \rangle$ for the duality pairing between tempered distributions $S \in \mathscr{S}'$ and rapidly decreasing functions $\phi \in \mathscr{S}$. Using the identity $\langle M, \phi \rangle = \langle \hat{M}, \hat{\phi}(-\,\cdot) \rangle$ and the Hermite–Genocchi formula again we can see that

$$\langle M, \phi \rangle = \int_{S^n} \phi\left(\sum_{\nu=0}^{n} \lambda_\nu z_\nu \right) d\lambda_1 \cdots d\lambda_n. \tag{2}$$

This relation was used by Micchelli to define the multivariate B-splines. From (2) the equivalence to de Boor's definition easily follows. In particular we have for $k+1$ affinely independent points $z_\nu \in \mathbb{R}^k$

$$\begin{aligned} M(x \mid z_0, \ldots, z_k) &= 1/(k! \operatorname{vol}_k(\operatorname{conv}\{z_0, \ldots, z_k\})) \\ &\qquad \text{for} \quad x \in \operatorname{conv}\{z_0, \ldots, z_k\} \\ &= 0 \qquad \text{otherwise.} \end{aligned} \tag{3}$$

Using the recurrence relation for divided differences

$$\begin{aligned} (t_\nu - t_\mu)[t_0, \ldots, t_n] = {} & [t_0, \ldots, t_{\mu-1}, t_{\mu+1}, \ldots, t_n] \\ & - [t_0, \ldots, t_{\nu-1}, t_{\nu+1}, \ldots, t_n] \end{aligned} \tag{4}$$

we can derive from (1) a formula for the directional derivative of M. We set

$t_\rho = -iy \cdot z_\rho$, apply (4) to the function e^t and take the inverse Fourier transform. It follows that

$$D_{z_\nu - z_\mu} M(x \mid z_0, \ldots, z_n) = M(x \mid z_0, \ldots, z_{\nu-1}, z_{\nu+1}, \ldots, z_n) - M(x \mid z_0, \ldots, z_{\mu-1}, z_{\mu+1}, \ldots, z_n). \tag{5}$$

Since $D_{\Sigma \lambda_\rho z_\rho} = \Sigma \lambda_\rho D_{z_\rho}$, (5) generalizes as follows.

THEOREM [4, 10]. *If* $y = \sum_{\nu=0}^n \lambda_\nu z_\nu$, $\sum_{\nu=0}^n \lambda_\nu = 0$, *then we have*

$$D_y M(x \mid z_0, \ldots, z_n) = \sum_{\nu=0}^n \lambda_\nu M(x \mid z_0, \ldots, z_{\nu-1}, z_{\nu+1}, \ldots, z_n). \tag{6}$$

Since the support of $M(x \mid z_{\nu_1}, \ldots, z_{\nu_k})$ is contained in a hyperplane, we conclude from (6) that M is a polynomial of total degree $n-k$ in every region not cut by one of the hyperplanes determined by a subset of k knots. If the knots $z_\nu \in \mathbb{R}^k$ are in general position, i.e., every subset of $k+1$ points is affinely independent, then $M(x \mid z_0, \ldots, z_n)$ globally belongs to C^{n-k-1}. In general M is a piecewise polynomial of class C^{n-l} if each subset of l knots forms a proper convex set [11].

By methods of Fourier analysis we now give another proof for the recurrence relations first obtained in [10]. For nonzero vectors $z_\nu \in \mathbb{R}^k$ let $H(z_1, \ldots, z_n)$ denote a tempered fundamental solution for the differential operator $\prod_{\nu=1}^n D_{z_\nu}$, i.e., $(\prod_\nu D_{z_\nu}) H = \delta$ [9]. On the set $\Omega_H = \{ y \in \mathbb{R}^k \mid \prod_{\nu=1}^n (iy \cdot z_\nu) \neq 0 \}$ the Fourier transform of H can be identified with the function

$$I(y \mid z_1, \ldots, z_n) = \prod_{\nu=1}^n (iy \cdot z_\nu)^{-1}. \tag{7}$$

For simplicity suppose that the points $z_0, \ldots, z_n$ are pairwise distinct. Then by expanding the divided difference in (1) we can see that

$$\hat{M}(y \mid z_0, \ldots, z_n) = \sum_{\nu=0}^n e^{-iy \cdot z_\nu} I(y \mid z_0 - z_\nu, \ldots, z_{\nu-1} - z_\nu, z_{\nu+1} - z_\nu, \ldots, z_n - z_\nu). \tag{8}$$

Thus, formally applying the inverse Fourier transform, we have formally obtained W. Dahmen's truncated power representation [4]. In [4] he explicitly constructs natural fundamental solutions H by repeated directional integration. But this will not be necessary for developing the recurrence relations. We first prove a slightly weaker version of a result in [4].

LEMMA. *Given $n > k$ nonzero vectors z_ν which span $\mathbb{R}^k$ and any set of affine functions λ_ν satisfying $x = \sum_\nu \lambda_\nu(x) z_\nu$ we have*

$$\langle (n-k)\, H(z_1,\dots,z_n), \phi\rangle = \left\langle \sum_{\nu=1}^{n} \lambda_\nu H(z_1,\dots,z_{\nu-1}, z_{\nu+1},\dots,z_n), \phi \right\rangle \tag{9}$$

for all $\phi \in \mathscr{S}$ with $\operatorname{supp} \hat{\phi} \subset \Omega_H$.

We have assumed the affinity of the functions λ_ν merely because we want to multiply them with the tempered distributions H.

Proof. From the definition of H it follows that

$$D_{z_\nu} H(z_1,\dots,z_n) = H(z_1,\dots,z_{\nu-1}, z_{\nu+1},\dots,z_n)$$

and, since $x = \Sigma\lambda_\nu z_\nu$, we obtain

$$\sum_{\nu=1}^{n} \lambda_\nu D_{z_\nu} H = D_x H.$$

Therefore, to complete the proof we have to show that H satisfies Euler's differential equation for a homogeneous distribution of degree $n-k$, i.e.,

$$\langle (n-k)\, H, \phi\rangle = \langle D_x H, \phi\rangle \tag{10}$$

on the restricted space of test functions.

Since $\hat{H}$ agrees on Ω_H with $I(y \mid z_1,\dots,z_n)$, a homogeneous function of degree $-n$, it follows that

$$\langle n\hat{H}, \psi\rangle = -\langle D_y \hat{H}, \psi\rangle.$$

Writing the right hand side in the form

$$\langle k\hat{H}, \psi\rangle + \sum_{\nu=1}^{k} \langle i\partial_\nu(iy_\nu \hat{H}), \psi\rangle$$

we obtain Eq. (10) by Fourier transformation. We set $\psi = \hat{\phi}(-\,\cdot)$ and use the identity $\langle S, \phi\rangle = \langle \hat{S}, \psi\rangle$.

Remark. For a homogeneous distribution H we get immediately from Euler's equation the sharper statement

$$(n-k)\, H(z_1,\dots,z_n) = \sum_{\nu=1}^{n} \lambda_\nu H(z_1,\dots,z_{\nu-1}, z_{\nu+1},\dots,z_n). \tag{9'}$$

In particular this is true for the homogeneous fundamental solutions constructed in [4]. But, it does not follow from (8) that we may express the

B-spline as a sum of homogeneous fundamental solutions. However, for $\phi \in \mathscr{S}$ with $\operatorname{supp} \hat{\phi} \subset \Omega_H$ we have from (8) in the case of pairwise distinct knots that

$$\langle M(\cdot \mid z_0,..., z_n), \phi \rangle$$
$$= \left\langle \sum_{r=0}^{n} T_{z_r} H(z_0 - z_r,..., z_{r-1} - z_r, z_{r+1} - z_r,..., z_n - z_r), \phi \right\rangle \quad (11)$$

no matter which particular set of fundamental solutions H we choose. Here T_{z_r} denotes a translation by the vector z_r, i.e., $\langle T_{z_r} S, \phi \rangle = \langle S, \phi(\cdot + z_r) \rangle$.

Similarly as in [4] Eq. (11) and the previous Lemma immediately lead to recurrence relations.

THEOREM. *Assume that the knots* $z_0,..., z_n$ span $\mathbb{R}^k$ *and*

$$x = \sum_{r=0}^{n} \lambda_r z_r, \qquad \sum_{r=0}^{n} \lambda_r = 1.$$

Then we have for $n > k$

$$M(x \mid z_0,..., z_n) = \frac{1}{n-k} \sum_{r=0}^{n} \lambda_r M(x \mid z_0,..., z_{r-1}, z_{r+1},..., z_n) \qquad (12)$$

if all B-*splines occurring in this equation are continuous at* x.

Note that we do not require the knots to be pairwise distinct. This theorem was first obtained by Micchelli [10], cf. also [11] for another proof. Under more restrictive assumptions on the knots and the coefficients λ_r it was also proved by Dahmen [4].

Proof. Since the knots $z_0,..., z_n$ are affinely independent, we can extend the constants λ_r to affine functions $\lambda_r(z)$ such that for all $z \in \mathbb{R}^k$ $z = \sum_{r=0}^{n} \lambda_r(z)$ and $\sum_{r=0}^{n} \lambda_r(z) = 1$. It is then sufficient to prove (12) in the distributional sense.

We first assume that the knots z_r are pairwise distinct and apply the Lemma to each of the distributions H occurring on the right hand side of (11). To this end we observe that since $\Sigma \lambda_r(x) = 1$, we have $x - z = \Sigma \lambda_r(x)(z_r - z)$. Hence, in particular,

$$x - z_u = \sum_{r \neq u} \lambda_r(x)(z_r - z_u),$$

i.e.,

$$\sum_{\substack{u=0 \\ u \neq r}}^{n} (T_{-z_r} \lambda_u)(x)(z_u - z_r) = x.$$

Therefore we obtain for $\phi \in \mathscr{S}$ with $\operatorname{supp} \hat{\phi} \subset \Omega_H$

$$\begin{aligned}\langle (n-k)\, M(\cdot \mid z_0,..., z_n), \phi \rangle &= \Big\langle \sum_{\nu=0}^{n} T_{z_\nu} \sum_{\substack{\mu=0\\ \mu\neq\nu}}^{n} (T_{-z_\nu}\lambda_\mu)\, H(z_0 - z_\nu,..., z_{\mu-1} - z_\nu, z_{\mu+1} \\ &\qquad - z_\nu,..., z_{\nu-1} - z_\nu, z_{\nu+1} - z_\nu,..., z_n - z_\nu), \phi \Big\rangle.\end{aligned}$$

Interchanging the order of summation and applying (11) we conclude that

$$\langle (n-k)\, M(\cdot \mid z_0,..., z_n), \phi \rangle = \Big\langle \sum_{\mu=0}^{n} \lambda_\mu M(\cdot \mid z_0,..., z_{\mu-1}, z_{\mu+1},..., z_n), \phi \Big\rangle. \qquad (13)$$

But, since the Fourier transforms of the B-splines are smooth and $\{\hat{\phi} \in \mathscr{S} \mid \operatorname{supp} \hat{\phi} \subset \Omega_H\}$ is L_1-dense in $\mathscr{S}$ we can drop the restriction on the test functions.

We now include the case of coalescing knots by continuity arguments. To this end we choose an approximating sequence $z_\nu(\varepsilon)$, $\varepsilon \to 0$, of pairwise distinct knots and corresponding affine functions $\lambda_\nu(x, \varepsilon)$. We may assume that for fixed $\phi \in \mathscr{S}$ the mapping $\varepsilon \to \lambda_\nu(\cdot, \varepsilon)\, \phi(\cdot)$ is continuous with respect to the topology of $\mathscr{S}$. If we can show the continuity of the mappings

$$\varepsilon \longmapsto \langle M(\cdot \mid z_0(\varepsilon),..., z_n(\varepsilon)), \phi \rangle,$$
$$\varepsilon \longmapsto \langle M(\cdot \mid z_0(\varepsilon),..., z_{\nu-1}(\varepsilon), z_{\nu+1}(\varepsilon),..., z_n(\varepsilon)), \lambda_\nu(\cdot, \varepsilon)\, \phi \rangle$$

we can pass to the limit in (13) and the theorem is proved. But, this is almost clear by taking Fourier transforms. To see this, e.g., in the second case, we write

$$\langle M_\varepsilon, \lambda_\varepsilon \phi \rangle - \langle M, \lambda\phi \rangle = \langle \hat{M}_\varepsilon(-\,\cdot), \widehat{\lambda_\varepsilon \phi} - \widehat{\lambda\phi} \rangle + \langle (\hat{M}_\varepsilon - \hat{M})(-\,\cdot), \widehat{\lambda\phi} \rangle.$$

This expression converges to zero because by the definition (1) of the B-splines we have $\|\hat{M}\|_\infty \leqslant 1$ and $\hat{M}_\varepsilon(y) \to \hat{M}(y)$.

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